

Minimal free resolution of a graded ideal with linear quotients*

A-Ming Liu[†] and Tongsuo Wu[‡]

Department of Mathematics, Shanghai Jiaotong University

Abstract. Let I be a graded ideal of $K[x_1, \dots, x_n]$ generated by homogeneous polynomials of a same degree d , and assume that I has linear quotients. In this note, we use Horseshoe Lemma to give a relatively direct inductive construction of a minimal free resolution of I , which is called a d -linear resolution.

Key Words: Graded ideal; linear quotients; d -linear resolution; Horseshoe Lemma

2010 AMS Classification: 13F20, 05E40

1 Preliminaries

We begin by recalling the following definition:

Definition 1.1. Let I be a graded ideal of S . If there exists a sequence of homogeneous generators f_1, \dots, f_m of I such that for any i with $1 < i \leq m$, the colon ideal

$$\langle f_1, f_2, \dots, f_{i-1} \rangle : f_i$$

is generated by linear forms, then I is said to have **linear quotients** (with respect to the ordering f_1, \dots, f_m).

Note that if a graded ideal I is generated by linear forms, then I has linear quotients. In fact, let l_1, \dots, l_r be a minimal generating set of linear forms of the ideal I . We claim

*This research is supported by the National Natural Science Foundation of China (Grant No. 11271250).

[†]aming8809@163.com

[‡]Corresponding author. tswu@sjtu.edu.cn

that l_1, \dots, l_r is a regular sequence of S . In fact, the minimality ensures that it is an \mathbb{K} -independent subset of S_1 . Then l_1, \dots, l_r can be extended to a \mathbb{K} -basis, say l_1, \dots, l_n , of S_1 . Then $x_i \mapsto l_i$, ($\forall i = 1, \dots, n$) induces an automorphism of the \mathbb{K} -algebra S . Since x_1, \dots, x_n is a regular sequence of S , clearly l_1, \dots, l_n is also a regular sequence of S . In particular, l_1, \dots, l_r is a regular sequence, hence

$$\langle l_1, l_2, \dots, l_{i-1} \rangle : l_i = \langle l_1, l_2, \dots, l_{i-1} \rangle, \forall 2 \leq i \leq r.$$

Thus I has linear quotients. In the case, note that $r = ht(I)$.

Recall that a polymatroidal monomial ideal has linear quotients (see, e.g., [5, Lemma 1.3]). For a simplicial complex Δ , recall that the Stanley-Reisner ideal I_Δ has linear quotients iff the Alexander dual complex Δ^\vee of Δ is shellable ([4, Proposition 8.2.5]).

Now assume that a graded ideal I has linear quotients with respect to the ordering f_1, \dots, f_m . As in [3], for each $2 \leq i \leq m$, let $q_i(I)$ be the cardinal number of a minimal generating set of linear forms of the colon ideal

$$\langle f_1, f_2, \dots, f_{i-1} \rangle : f_i.$$

Then it follows from the previous paragraph that $q_i(I)$ is independent of the choice of a minimal generating set of linear forms of the colon ideal. Let

$$q(I) = \max\{q_i(I) \mid 2 \leq i \leq m\}.$$

Note that if I is generated by linear forms, then $q_i(I) = i - 1$, thus $q(I) = r - 1$, where r is the cardinal number of a minimal generating set of linear forms of I .

For a graded ideal I of the graded ring S , let

$$0 \longrightarrow \oplus_{i=1}^{b_r} R(-d_{ri}) \longrightarrow \dots \longrightarrow \oplus_{i=1}^{b_1} R(-d_{1i}) \longrightarrow I \longrightarrow 0$$

be the minimal graded resolution of I by free modules. The ideal I is said to have a *pure resolution* if there are constants $d_1 < d_2 < \dots < d_r$, such that

$$d_{1i} = d_1, \dots, d_{ri} = d_r, \forall i.$$

If further $d_1 = d$, $d_i = d_1 + i - 1$, $\forall 2 \leq i \leq r$, then I is said to *admit a d -linear resolution*. See [1, Theorem 4.3.1] for a characterization given by Eisenbud and Goto. For a simplicial complex Δ , recall that the ideal I_Δ admits a linear resolution if and only if the complex Δ^\vee is Cohen-Macaulay over any field K (Eagon-Reiner Theorem, see [4, Theorem 8.1.9]).

2 A construction of minimal free resolution by using Horseshoe Lemma

Lemma 2.1. *Let I be a graded ideal of S generated by linear forms. Then I has 1-linear resolution and $pd_S(I) = q(I)$.*

Proof. Let f_1, \dots, f_r be a minimal generating set of linear forms of the ideal I and denote $I_t = \langle f_1, \dots, f_t \rangle, 1 \leq t \leq r$. Then $q_t(I) = q(I_t) = t - 1$.

We prove the result by using induction on r . For $r = 1$, the result is clear. Now assume $r > 1$ and consider the exact sequence of graded S -modules and graded S -module homomorphisms

$$0 \longrightarrow I_{r-1} \longrightarrow I_r \longrightarrow I_r/I_{r-1} \longrightarrow 0.$$

By induction, assume that I_{r-1} has the following 1-linear free resolution of projective dimension $r - 2$:

$$0 \longrightarrow F_{r-2} \xrightarrow{\sigma_{r-2}} \dots \longrightarrow F_0 \xrightarrow{\sigma_0} I_{r-1} \longrightarrow 0.$$

Note that $I_r/I_{r-1} = S\overline{f_k}$, thus we have the following graded exact sequence

$$0 \longrightarrow I_{r-1}(-1) \longrightarrow S(-1) \longrightarrow I_r/I_{r-1} \longrightarrow 0,$$

hence the following is the 1-linear free resolution of I_r/I_{r-1} of projective dimension $r - 1$:

$$0 \longrightarrow F_{r-2}(-1) \xrightarrow{\sigma_{r-2}} \dots \longrightarrow F_0(-1) \xrightarrow{\sigma_0} S(-1) \xrightarrow{\sigma'_0} I_r/I_{r-1} \longrightarrow 0.$$

We then use the construction in proving the Horseshoe Lemma in homological algebra (see, e.g., [6, Proposition 6.24]), to construct a free resolution of I_r with $pd_S(I_r) = r - 1$:

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
0 & \longrightarrow & 0 & \xrightarrow{j} & 0 \oplus F_{r-2}(-1) & \xrightarrow{\pi} & F_{r-2}(-1) \longrightarrow 0 \\
& & \downarrow & & \downarrow \delta_{r-1} & & \downarrow \sigma_{r-2} \\
0 & \longrightarrow & F_{r-2} & \xrightarrow{j} & F_{r-2} \oplus F_{r-3}(-1) & \xrightarrow{\pi} & F_{r-3}(-1) \longrightarrow 0 \\
& & \sigma_{r-2} \downarrow & & \downarrow \delta_{r-2} & & \downarrow \sigma_{r-3} \\
& & \vdots & & \vdots & & \vdots \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & F_1 & \xrightarrow{j} & F_1 \oplus F_0(-1) & \xrightarrow{\pi} & F_0(-1) \longrightarrow 0 \\
& & \sigma_1 \downarrow & & \downarrow \delta_1 & & \downarrow \sigma_0 \\
0 & \longrightarrow & F_0 & \xrightarrow{j} & F_0 \oplus S(-1) & \xrightarrow{\pi} & S(-1) \longrightarrow 0 \\
& & \sigma_0 \downarrow & & \downarrow \delta_0 & & \downarrow \sigma'_0 \\
0 & \longrightarrow & I_{r-1} & \longrightarrow & I_r & \longrightarrow & I_r/I_{r-1} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & 0 & & 0 & & 0
\end{array}$$

By the construction, each δ_i is essentially the sum of σ_i and σ'_i , thus the constructed middle resolution is a 1-linear resolution of I_r . This shows that I has 1-linear resolution and $pd_S(I) = r - 1 = q(I)$ ■

Again let $I_t = \langle f_1, \dots, f_t \rangle, 1 \leq t \leq r$. Let $L_k = \langle f_1, \dots, f_{k-1} \rangle : f_k$ be the series of colon ideals. By Lemma 2.1,

$$pd_S(S/L_k) = pd_S(L_k) + 1 = q_k(I).$$

Note that

$$0 \longrightarrow I_{k-1} \longrightarrow I_k \longrightarrow (S/L_k)(-d) \longrightarrow 0$$

is a exact sequence of graded modules, thus

$$pd(I_k) = \max\{pd(I_{k-1}), q_k(I)\} = q(I_k).$$

So, if we use the previous lemma and the proof to the above sequences, then in a similar manner we use induction and the Horseshoe Lemma to give a relatively direct proof to the following:

Theorem 2.2. (*[4, Proposition 8.2.1 and Corollary 8.2.2]*) *Let I be a graded ideal of S generated by homogeneous polynomials of degree d . If I has linear quotients, then I has a d -linear resolution and, $pd_S(I) = q(I)$.*

References

- [1] Bruns W. and Herzog J.. *Cohen-Macaulay Rings*. **Cambrisse University Press**, Cambridge, Rev. Ed., 1997.
- [2] Eisenbud D. *Commutative Algebra with a View Toward Algebraic Geometry*. **Springer Science + Business Media, Inc.** 2004.
- [3] Herzog J. and Hibi T. Cohen-Macaulay polymatroidal ideals, *European Journal of Combinatorics* 27(2006) 513 – 517.
- [4] Herzog J. and Hibi T. *Monomial Ideals*. **Springer-Verlag London Limited**, 2011.
- [5] J. Herzog, Y. Takayama Resolutions by mapping cones, *Homology Homotopy Appl.* 4(2002) 277 – 294.
- [6] Rotman J.J. *An Introduction To Homological Algebra*. **Springer Science +Business Media, LLC** 2009. Second Edition.